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Monopole emission of sound by asymmetric bubble oscillations. Part 1. Normal modes

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On a linearized theory, the pressure field due to bubbles oscillating asymmetrically in a 'distortion mode' decays with radial distance r like $r^{-(n+1)}$, where $n > 1$. Hence these modes have been thought to produce a negligible emission of sound. In this paper it is shown that, on the contrary, in nonlinear theory the distortion modes produce a monopole radiation of sound ($n = 0$) at second order. Its frequency is twice the basic frequency of the distortion mode, and the sound amplitude is proportional to the square of the distortion amplitude. The magnitude of the pressure fluctuations within the bubble is comparable with 1 atmosphere.

1. Introduction

The production of sound by a rippling stream, or by a water surface lightly ruffled by the wind, or by jets entering a free surface, or by air bubbles emerging from an underwater nozzle, are all phenomena which despite their familiarity are far from being fully understood. The sound of bubbles forming at a nozzle was first studied and partly explained by Minnaert (1933) and afterwards by Meyer & Tamm (1943). Minnaert showed that the frequency of the sound was experimentally close to that of the radial mode of oscillation of a spherical bubble containing the observed volume of air – the so-called 'breathing mode'. His demonstration relied on a simple, linear theory of small pulsations, in which the kinetic energy in the water was equated to the extra potential energy stored in the bubble gas.

Fully nonlinear theories for such oscillations have been given by Nottingk & Neppiras (1950) and many later workers; for a review see Plesset & Prosperetti (1977). The emphasis in such studies has usually been on the explanation of cavitation phenomena. On the other hand, for the sound produced by bubbles near a free surface we are concerned with the oscillation of air bubbles at approximately atmospheric pressure.

All of the above work leaves unanswered a fundamental question: how are the bubble oscillations set in motion? It seems to have been tacitly assumed that the motion was due to the addition of a sudden ambient pressure immediately following the closure of the bubble surface. At a recent conference on underwater sound from the sea surface both Crowther (1987) and Hollett & Heitmeyer (1987) showed that an additional static pressure of say 10 cm of water at the instant of bubble formation

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might be sufficient to explain the observed sound intensity from a breaking wave. In the present paper, however, we wish to draw attention to the much larger effects probably produced by the initial *distortion* of the bubbles, both in breaking waves and in other circumstances of bubble formation.

The distorted, or asymmetric, oscillations of bubbles have usually been dismissed as significant sources of sound (see for example Fitzpatrick & Strasberg 1957) on the grounds, first, that the natural frequency was too low, and second that on the linear theory, as given by Lamb (1932, p. 475), both the motion and the associated pressure fluctuations decay too rapidly with radial distance r from the bubble. Unlike the breathing mode, which creates a pressure field varying like r^{-1} (a monopole), the 'distortion modes' decay like higher inverse powers of r. A *non*linear theory for distortion modes has been given by Tsamopoulos & Brown (1983) which, however, appears incomplete (see §6 below).

Now a simple analogy with standing waves on deep water suggests that there could well be second-order, or higher, nonlinear terms in the pressure fields of a distortion mode which are not attenuated so rapidly with r as are the first-order terms. To study these we must extend the linear theory to higher order. In this paper we carry out such an investigation and find that at second order the distortion modes do indeed emit a monopole radiation, varying only like r^{-1}. The radiation has a frequency double that of the linear oscillation, and an amplitude proportional to the square of distortion amplitude. A first glance at orders of magnitude shows that the amplitude is surprisingly large: in and near the bubble itself, the pressure changes can be of the order of one atmosphere or more.

The orders of magnitude are discussed first in §2 below. In §§3 and 4 we derive the full, nonlinear boundary conditions at the surface of the bubble, and in §5 we derive equations in a perturbation scheme, correct to second order. There is a simplification, in that at second order it is necessary to consider only the equations for the *spherically averaged* motion and pressure field, the remainder of the motion being relatively small when r is large. The axisymmetric normal modes are discussed in detail in §§6 and 7.

The analogy with standing waves on the surface of deep water is described briefly in §8. This leads, in §9, to the discussion of a different type of oscillation of a spherical bubble, namely an eastwards or westwards travelling wave. When, and only when, the eastwards and westwards waves are present simultaneously, do they emit a monopole radiation, which is proportional to the product of their amplitudes. Conclusions follow in §10.

2. Orders of magnitude

To define the range of bubble sizes in which we are interested we show in figure 1 the range of diameters for bubbles observed visually near the surface of a wind-wave channel, at different wind speeds, by Toba (1961). His diagram is reproduced here for convenience. At the highest wind speeds the diameters range from about 0.3 to 10 mm, hence the radii a lie between 0.015 and 0.5 cm. This is certainly much larger than the bubble sizes found at depths of 1 m or more in the ocean. The range of sizes may be compared with the bubble radii studied by Minnaert (1933): 0.17 to 0.30 cm, and by Fitzpatrick & Strasberg (1957): $a = 0.23$ cm. For the purpose of discussion we shall assume that

$$0.01 \text{ cm} \leqslant a \leqslant 1.0 \text{ cm}. \tag{2.1}$$

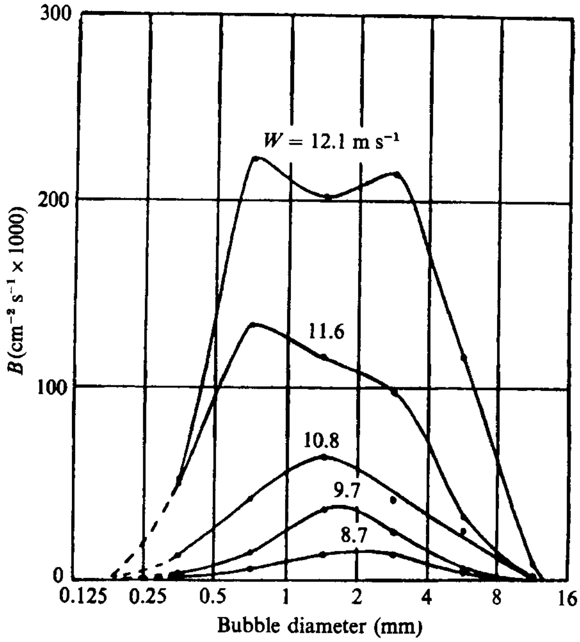

FIGURE 1. Calculated mean rate B of bubble production at the sea surface, per log (bubble radius) (from Toba 1961).

Now in the well-known linear theory of bubble oscillations (see for example Plesset & Prosperetti 1977) the radian frequency ω of the fundamental radial mode is given by

$$\omega^2 = \frac{3\gamma p_0}{\rho a^2} - \frac{2T}{\rho a^3}, \tag{2.2}$$

where γ is the ratio of the specific heats ($= 1.4$ for adiabatic changes in air), p_0 is the equilibrium pressure and a the equilibrium radius of the bubble, ρ the density of water and T surface tension. In what follows we shall generally set $\rho = 1$ g/cm and $T = 75$ dyne/cm. Moreover, if z denotes the depth of the bubble below the free surface, and p_{A} the atmospheric pressure, then

$$p_0 = p_{\mathrm{A}} + \rho g z + \frac{2T}{a}. \tag{2.3}$$

The atmospheric pressure being somewhat less than 10^6 dyne/cm^2, and a being at least 0.01 cm it is clear that both the second and third terms on the right of (2.3) are small compared to the first, and we may take

$$p_0 = 1.0 \times 10^6 \text{ dyne/cm}^2 \tag{2.4}$$

to a fair approximation. Thus for bubble sizes in the range (2.1) the radian frequency lies between 2×10^3 s^{-1} and 2×10^5 s^{-1}, corresponding to frequencies $\omega/2\pi$ lying between 0.3 and 30 kHz, roughly.

The asymmetric modes of oscillation, on the other hand, have radian frequencies σ_n given by

$$\sigma_n^2 = (n-1)(n+1)(n+2)\frac{T}{a^3}, \tag{2.5}$$

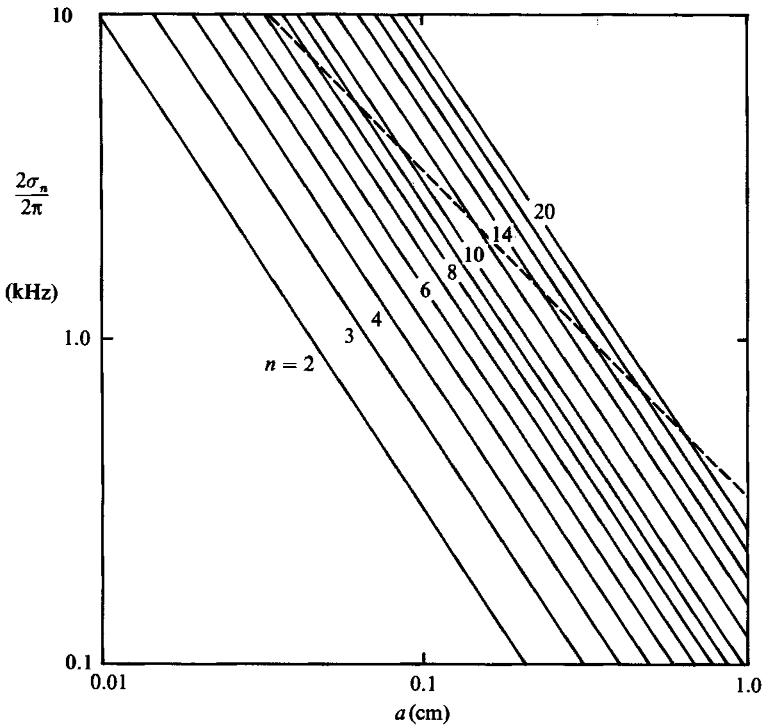

FIGURE 2. Radian frequency $2\sigma_n$ of the second harmonic in a distortion mode of degree n, compared to the frequency ω of the breathing mode (broken line).

n	a (cm)	$\omega/2\pi$ (kHz)	n	a (cm)	$\omega/2\pi$ (kHz)
2	0.00074	472.0	9	0.0627	5.20
3	0.00274	121.0	10	0.0847	3.85
4	0.00631	57.1	11	0.111	2.93
5	0.0119	27.6	12	0.143	2.28
6	0.0199	16.5	13	0.180	1.81
7	0.0307	10.6	14	0.223	1.46
8	0.0450	7.28	15	0.272	1.20

TABLE 1. Bubble radius at which $2\sigma_n = \omega$

where n is the order of the corresponding spherical harmonic (see Lamb 1932). The first-order (linearized) pressure fluctuations in this type of oscillation decay like $r^{-(n+1)}$ with radial distance r from the centre of the bubble, and so become negligible even at moderate distances, when $n > 1$.

Consider now the doubled frequency $2\sigma_n$. The situation is seen more clearly in figure 2 where $2\sigma_n$ and ω are plotted as functions of the bubble radius a. It will be seen that there are resonances between $2\sigma_n$ and ω depending on the value of a. Table 1 below lists the values of a and the corresponding values of n and $2\sigma_n$, or ω. Most of these lie within the range of interest.

It is also interesting to consider the order of magnitude of the second-order pressure fluctuations. Suppose we have an initial distortion ϵa of the bubble surface from its spherical shape, where ϵ is the relative distortion. The particle velocities accompanying the oscillation will be of order $\epsilon a \sigma_n$, and the changes in pressure, from Bernoulli's equation, will be of order

$$p' = \tfrac{1}{2}(\epsilon a \sigma_n)^2. \tag{2.6}$$

Comparing this with the equilibrium pressure p_0 in the bubble we have from (2.2)

$$\frac{p'}{p_0} = \tfrac{3}{2}\gamma\epsilon^2\frac{\sigma_n^2}{\omega^2} = 0.525\epsilon^2\left(\frac{2\sigma_n}{\omega}\right)^2. \tag{2.7}$$

At resonance, when $2\sigma_n = \omega$, and with ϵ as small as 0.5 we get pressure changes equivalent to about one eighth of an atmosphere, that is to say local fluctuations as great as 1.25 m of water. Even when $\epsilon = 0.2$ we still get 20 cm of water, which is greater than the pressure changes (10 cm of water) assumed by Hollett & Heitmeyer (1987) in their theory of sound generation by sudden applications of pressure to bubbles in a spilling breaker. Since much larger relative distortions of the bubbles are readily conceivable, it appears that here we may have a very significant mechanism for the generation of underwater sound.

It remains to be shown that such second-order asymmetric oscillations can produce oscillations at a distance r which vary only like r^{-1}, as in the symmetric radial oscillation. This we shall now demonstrate.

3. Boundary conditions (1)

Since the bubble radii to be considered are much smaller than the wavelength of the emitted sound, it is permissible when discussing the motion near the bubble itself, to assume that the compressibility of the water can be neglected. We also assume initially that the flow is irrotational and that viscosity can be neglected.

We adopt radial coordinates r, θ, ϕ as shown in figure 3, and write the equation of the bubble surface as

$$r = R(\theta, \phi, t), \tag{3.1}$$

where t denotes the time. Then the *kinematic* boundary condition can be written as

$$\frac{\mathrm{D}}{\mathrm{D}t}(R-r) = 0, \tag{3.2}$$

where $\mathrm{D}/\mathrm{D}t$ denotes differentiation following the motion:

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}, \tag{3.3}$$

$\boldsymbol{u}$ being the particle velocity. The vorticity being neglected, we may write

$$\boldsymbol{u} = \boldsymbol{\nabla}\phi, \tag{3.4}$$

where $\Phi(r, \theta, \phi, t)$ is the velocity potential. Then (3.2) becomes

$$R_t + \left(\Phi_r, \frac{\Phi_\theta}{r}, \frac{\Phi_\phi}{r\sin\theta}\right)\cdot\left(-1, \frac{R_\theta}{r}, \frac{R_\phi}{r\sin\theta}\right) = 0. \tag{3.5}$$

Here subscripts are used to denote partial differentiation and a dot $(\cdot)$ denotes the inner vector product. Hence we have

$$R_t - \Phi_r = -\frac{1}{r^2}\left(R_\theta\Phi_\theta + \frac{R_\phi\Phi_\phi}{\sin^2\theta}\right), \tag{3.6}$$

to be satisfied when $r = R$.

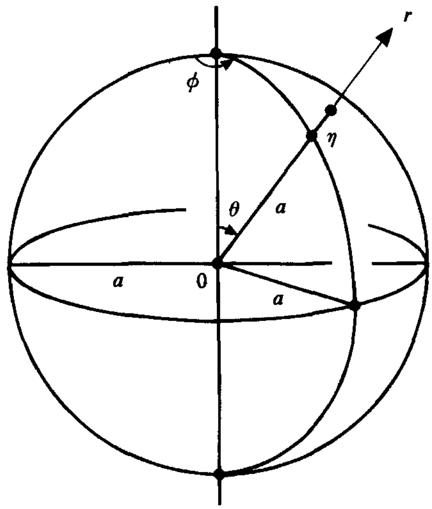

FIGURE 3. Definition of spherical coordinates.

It is convenient to replace (3.6) by an equivalent condition to be satisfied on the equilibrium surface $r = a$. We do this by expanding each side of (3.6) in a Taylor series about $r = a$. Hence if we write

$$R = a+\eta(\theta,\phi,t) \tag{3.7}$$

we find that, correct to second order in η,

$$\eta_t-\Phi_r = \eta\Phi_{rr}-\frac{1}{a^2}\left(\eta_\theta\Phi_\theta+\frac{\eta_\phi\Phi_\phi}{\sin^2\theta}\right) \tag{3.8}$$

is to be satisfied when $r = a$.

4. Boundary conditions (2)

If we neglect viscous terms as before, then the dynamical boundary condition to be satisfied when $r = R(\theta,\phi,t)$ is that the pressures on the two sides of the surface differ only because of surface tension, i.e. if p and p_{B} denote the pressure in the water and in the bubble, respectively, and if we take the density of the water as unity, then

$$p_{\mathrm{B}} = p+T\nabla\cdot\boldsymbol{n}. \tag{4.1}$$

Here $\boldsymbol{n}$ denotes the unit normal to the surface $r = R$, and we use the theorem that $\nabla\cdot\boldsymbol{n}$ is equal to the sum of the curvatures, see Lamb (1932, p. 474).

Now at points within the water we have Bernoulli's equation

$$p+\Phi_t+\tfrac{1}{2}(\nabla\Phi)^2 = F(t). \tag{4.2}$$

Assuming that as $r\rightarrow\infty$, both Φ and its derivatives tend to 0, we see that $F(t)$ is simply equal to the pressure p_∞ at infinity. Hence (4.1) can be written

$$\Phi_t-T\nabla\cdot\boldsymbol{n} = (p_\infty-p_{\mathrm{B}})-\tfrac{1}{2}(\nabla\Phi)^2. \tag{4.3}$$

We now examine the curvature term more carefully. The direction of the normal is that of the vector

$$\nabla(r-R) = \left(1,\frac{-\eta_\theta}{r},\frac{-\eta_\phi}{r\sin\theta}\right), \tag{4.4}$$

and we have
$$|\nabla(r-R)| = \left[1+\frac{1}{r^2}\left(\eta_\theta^2+\frac{\eta_\phi^2}{\sin^2\theta}\right)\right]^{\frac{1}{2}} = A \tag{4.5}$$

say. Thus
$$\boldsymbol{n} = \frac{1}{A}\nabla(r-\eta) \tag{4.6}$$

and
$$\nabla\cdot\boldsymbol{n} = \nabla\left(\frac{1}{A}\right)\cdot\nabla(r-\eta)+\frac{1}{A}\nabla^2(r-\eta). \tag{4.7}$$

But
$$\nabla\left(\frac{1}{A}\right)\cdot\nabla(r-\eta) = -\frac{1}{r^3}\left(\eta_\theta^2+\frac{\eta_\phi^2}{\sin^2\theta}\right)+O(\eta^3) \tag{4.8}$$

by (4.4) and
$$\nabla^2(r-\eta) = \frac{2}{r}-\frac{1}{r^2}\nabla_S^2\eta, \tag{4.9}$$

where ∇_S^2 denotes the surface Laplacian:
$$\nabla_S^2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \tag{4.10}$$

Also, by (4.5)
$$\frac{1}{A} = 1-\frac{1}{2r^2}\left(\eta_\theta^2+\frac{\eta_\phi^2}{\sin^2\theta}\right)+O(\eta^4), \tag{4.11}$$

and so correct to second order we obtain
$$\nabla\cdot\boldsymbol{n} = \frac{2}{r}-\frac{1}{r^2}\nabla_S^2\eta. \tag{4.12}$$

Now we wish to replace (4.12) by an expression to be evaluated on $r = a$. So we expand the powers of r in a Taylor series about $r = a$ to give finally
$$\nabla\cdot\boldsymbol{n} = \frac{2}{a}-\frac{1}{a^2}(2\eta+\nabla_S^2\eta)+\frac{1}{a^3}2\eta(\eta+\nabla_S^2\eta). \tag{4.13}$$

In (4.3) the term $\frac{1}{2}(\nabla\Phi)^2$ is already of second order in Φ, hence η, and if we write
$$p_B-p_\infty = \frac{2T}{a}+p_B', \tag{4.14}$$

where p_B' is now the pressure perturbation in the bubble, we have
$$\Phi_t+\frac{T}{a^2}(2+\nabla_S^2)\eta+p_B' = -\eta\Phi_{rt}-\tfrac{1}{2}(\nabla\Phi)^2+\frac{T}{a^3}2\eta(1+\nabla_S^2)\eta, \tag{4.15}$$

to be satisfied when $r = a$.

To complete the boundary condition (4.15) we need some assumption as to the gas pressure p_B. We take the static law
$$\frac{p_B}{p_0} = \left(\frac{\tau_0}{\tau}\right)^\gamma \tag{4.16}$$

where p_0 is the equilibrium pressure:
$$p_0 = p_\infty+\frac{2T}{a}, \tag{4.17}$$

τ_0 is the equilibrium volume:
$$\tau_0 = \tfrac{4}{3}\pi a^3, \tag{4.18}$$

and τ is the instantaneous volume:

$$\tau = \iint\left(a+\eta\right)^3 \sin\theta \,\mathrm{d}\theta\,\mathrm{d}\phi. \tag{4.19}$$

On expanding the integrand binomially we obtain

$$\frac{\tau}{\tau_0} = 1+\frac{3\overline{\eta}}{a}+\frac{3\overline{\eta^2}}{a^2}+\frac{\overline{\eta^3}}{a^3}, \tag{4.20}$$

where an overbar denotes integration over the unit sphere. So on substitution in (4.16) we have, correct to second order,

$$\frac{p_\mathrm{B}}{p_0} = 1-\frac{3\gamma\overline{\eta}}{a}-\frac{3\gamma[\overline{\eta^2}-(\gamma+1)\,\overline{\eta}^2]}{a^2}. \tag{4.21}$$

5. Small perturbations

Let us introduce the expansions

$$\left.\begin{aligned} \eta &= \eta_1+\eta_2+\dots, \\ \Phi &= \Phi_1+\Phi_2+\dots, \\ p_\mathrm{B} &= p_{\mathrm{B}1}+p_{\mathrm{B}2}+\dots, \end{aligned}\right\} \tag{5.1}$$

where η_m, Φ_m and p_m are all of order ϵ^m, ϵ being an ordering parameter. On substituting these expressions in the two boundary conditions (3.8) and (4.15), using (4.21) and equating coefficients of ϵ^m we find at first order ($m = 1$)

$$\eta_{1t}-\Phi_{1r} = 0 \tag{5.2}$$

and

$$\Phi_{1t}+\frac{T}{a^2}(2+\nabla_\mathrm{S}^2)\,\eta_1-\omega_0^2 a\overline{\eta}_1 = 0, \tag{5.3}$$

where

$$\omega_0^2 = \frac{3\gamma p_0}{a^2} = \omega^2+\frac{2T}{a^3}. \tag{5.4}$$

Also, from (4.2),

$$p_1 = -\Phi_{1t}. \tag{5.5}$$

These are well-known equations of linear theory. At second order ($m = 2$) we obtain

$$\eta_{2t}-\Phi_{2r} = \eta_1\Phi_{1rr}-\frac{1}{a^2}\left(\eta_{1\theta}\Phi_{1\theta}+\frac{\eta_{1\phi}\Phi_{1\phi}}{\sin^2\theta}\right) \tag{5.6}$$

and

$$\Phi_{2t}+\frac{T}{a^2}(2+\nabla_\mathrm{S}^2)\,\eta_2-\omega_0^2 a\overline{\eta}_2 = -\eta_1\Phi_{1rt}-\tfrac{1}{2}(\nabla\Phi_1)^2+\frac{T}{a^3}2\eta_1(1+\nabla_\mathrm{S}^2)\,\eta_1+\omega_0^2[\overline{\eta_1^2}-\tfrac{1}{2}(\gamma+1)\,\overline{\eta}_1^2]. \tag{5.7}$$

It will be seen that the operators on the left-hand sides of (5.6) and (5.7) are similar to those in (5.2) and (5.3), but that the right-hand sides are now quadratic, or rather bilinear, expressions in η_1 and Φ_1.

The equations for the mean, i.e. spherically averaged, quantities $\overline{\eta}_2$ and $\overline{\Phi}_2$ are of even simpler form. For we have in general

$$\overline{\nabla_\mathrm{S}^2\eta_2} = \nabla_\mathrm{S}^2\overline{\eta}_2 = 0 \tag{5.8}$$

and also
$$\Phi_{1r} = \eta_{1t} \tag{5.9}$$
by (5.2). Furthermore, for the distortion modes,
$$\bar{\eta}_1 = 0. \tag{5.10}$$
Thus on averaging both sides of (5.6) and (5.7) we have
$$\bar{\eta}_{2t} - \bar{\Phi}_{2r} = \overline{\eta_1 \Phi_{1rr}} - \frac{1}{a^2}\left(\overline{\eta_{1\theta}\Phi_{1\theta}} + \overline{\frac{\eta_{1\phi}\Phi_{1\phi}}{\sin^2\theta}}\right) \tag{5.11}$$
and
$$\bar{\Phi}_{2t} - \omega^2 a\bar{\eta}_2 = -\overline{\eta_1\eta_{1tt}} - \tfrac{1}{2}\overline{(\nabla\Phi_1)^2} + \frac{2T}{a^3}\overline{\eta_1(2+\nabla_S^2\eta_1)} + \omega^2\overline{\eta_1^2}, \tag{5.12}$$
where ω is given by (2.2). It will be found advantageous to define the modified displacement
$$h_2 = \bar{\eta}_2 + \overline{\eta_1^2}/a = \tfrac{1}{3}a(\tau/\tau_0 - 1) \tag{5.13}$$
by (4.20). Thus h_2 is proportional to the change in bubble volume. Equations (5.11) and (5.12) then become
$$h_{2t} - \bar{\Phi}_{2r} = \overline{\eta_1 \Phi_{1rr}} + \frac{2}{a}\overline{\eta_1\eta_{1t}} - \frac{1}{a^2}\left(\overline{\eta_{1\theta}\Phi_{1\theta}} + \overline{\frac{\eta_{1\phi}\Phi_{1\phi}}{\sin^2\theta}}\right) \tag{5.14}$$
and
$$\bar{\Phi}_{2t} - \omega^2 a h_2 = -\tfrac{1}{2}\overline{(\nabla\Phi_1)^2} - \overline{\eta_1\eta_{1tt}} + \frac{2T}{a^3}\overline{\eta_1(2+\nabla_S^2\eta_1} \tag{5.15}$$
respectively.

6. Asymmetric normal modes

Let us take
$$\eta_1 = a_n(t)\, S_n(\theta,\phi) \tag{6.1}$$
and
$$\Phi_1 = b_n(t)\left(\frac{a}{r}\right)^{n+1} S_n(\theta,\phi), \tag{6.2}$$
where $S_n(\theta,\phi)$ is a spherical harmonic of positive degree n (thus excluding symmetric oscillations) and a_n, b_n are functions of the time t only. Since Φ_1 is proportional to $r^{-(n+1)}$ the form (6.2) satisfies the equation of continuity $\nabla^2\Phi_1 = 0$, and also vanishes at infinity. We note that for any spherical harmonic S_n,
$$\nabla_S^2 S_n = -n(n+1)\, S_n, \tag{6.3}$$
where ∇_S^2 is given by (4.12). Moreover
$$\overline{S_n(\theta,\phi)} = 0, \quad n = 1, 2, \ldots, \tag{6.4}$$
so that $\bar{\eta}_1$ vanishes and there is no change in the volume of the bubble – at this order. Substituting into the boundary conditions (5.2) and (5.3) we have
$$\left.\begin{aligned} \frac{\mathrm{d}a_n}{\mathrm{d}t} &= -\frac{n+1}{a}b_n, \\ \frac{\mathrm{d}b_n}{\mathrm{d}t} &= (n-1)(n+2)\frac{T}{a^2}a_n. \end{aligned}\right\} \tag{6.5}$$

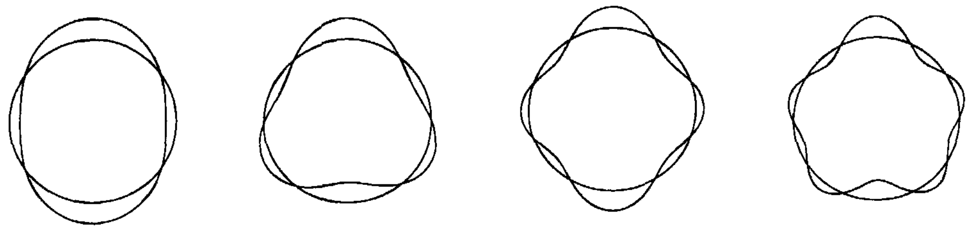

FIGURE 4. Axial sections of the axisymmetric normal modes: $n = 2$, 3, 4 and 5.

Hence all the conditions of the problem are satisfied by taking

$$\left.\begin{aligned} a_n &= A_n \cos\sigma_n t, \\ b_n &= \frac{a\sigma_n}{n+1} A_n \sin\sigma_n t, \end{aligned}\right\} \tag{6.6}$$

with
$$\sigma_n^2 = (n-1)(n+1)(n+2)\,T/a^3 \tag{6.7}$$

as stated in §2. The fluctuation p_1 in the pressure is given by

$$p_1 = -\Phi_{1t} = -\frac{a\sigma_n^2}{n+1} A_n \left(\frac{a}{r}\right)^{n+1} S_n(\theta,\phi)\cos\sigma_n t. \tag{6.8}$$

For axisymmetric oscillations, when $S_n(\theta,\phi)$ is taken to be independent of ϕ, we have
$$S_n(\theta,\phi) = P_n(\cos\theta), \tag{6.9}$$

where P_n denotes the Legendre polynomial of degree n. We have $P_0 = 1$ and $P_1 = \cos\theta$, the case $n = 1$ representing a pure bodily translation of the sphere, with $\sigma_1 = 0$. The simplest case for which $\sigma_n \neq 0$ is $n = 2$ for which we have

$$P_2(\cos\theta) = \tfrac{3}{2}\cos^2\theta - \tfrac{1}{2} \tag{6.10}$$

(see figure 4). In this mode the maximum radial displacement at each pole $\theta = 0$ or π is just twice that at the equator ($\theta = \frac{1}{2}\pi$).

Consider now the second approximation. On substituting η_1 and Φ_1 into (5.6), for example, we have
$$\eta_{2t} - \Phi_{2r} = \frac{a_n b_n}{a^2}\left[(n+1)(n+2)S_n^2 - \left(S_{n\theta}^2 + \frac{S_{n\phi}^2}{\sin^2\theta}\right)\right], \tag{6.11}$$

and so from (6.6) and (6.7)

$$\eta_{2t} - \Phi_{2r} = L(\theta,\phi)\sin 2\sigma_n t, \tag{6.12}$$

where $L(\theta,\phi)$ is of the form
$$L(\theta,\phi) = L' S_n^2 + L''\left(S_{n\theta}^2 + \frac{S_{n\phi}^2}{\sin^2\theta}\right) \tag{6.13}$$

and L, L'' denote constants independent of θ and ϕ. Similarly on subtituting into (5.7) we obtain an equation of the form

$$\Phi_{2t} + \frac{T}{a^3}(2+\nabla_S^2)\,\eta_2 - a\omega^2\bar{\eta}_2 = M(\theta,\,\phi)\cos 2\sigma_n t + N(\theta,\,\phi), \tag{6.14}$$

where $M(\theta,\phi)$ and $N(\theta,\phi)$ are of the same form as $L(\theta,\phi)$.

Now it is clear that S_n^2 and $S_{n\theta}^2$ are each polynomials in $\cos\theta$ of degree $2n$. So they may be expressed as the sum of Legendre polynomials P_m of degrees $m \leqslant 2n$. Hence one can express the right-hand sides of (6.12) and (6.14) in the forms

$$(U_0 S_0 + U_2 S_2 + \ldots + U_{2n} S_{2n}) \sin 2\sigma_n t \tag{6.15}$$

and

$$(V_0 S_0 + V_2 S_2 + \ldots + V_{2n} S_{2n}) \cos 2\sigma_n t + (W_0 S_0 + W_2 S_2 + \ldots + W_{2n} S_{2n}) \tag{6.16}$$

respectively. So one can find solutions to the boundary conditions in the forms

$$\eta_2 = (X_0 S_0 + X_2 S_2 + \ldots + X_{2n} S_{2n}) \cos 2\sigma_n t + (Y_0 S_0 + Y_2 S_2 + \ldots + Y_{2n} S_{2n}) \tag{6.17}$$

and
$$\Phi_2 = \left[Z_0 \frac{a}{r} S_0 + Z_2 \left(\frac{a}{r}\right)^3 S_2 + \ldots + Z_{2n} \left(\frac{a}{r}\right)^{2n+1} S_{2n} \right] \sin 2\sigma_n t \tag{6.18}$$

by suitable choice of the coefficients X_m, Y_m and Z_m. However, because of the high inverse powers of r in the expression for Φ_2, the behaviour of the motion at large distances r from the bubble will be dominated by the terms in $m = 0$. Since $S_0 = 1$ that implies

$$\Phi_2 \sim Z_0 \frac{a}{r} \sin 2\sigma_n t. \tag{6.19}$$

This represents a monopole source of sound, independent of the directions θ and ϕ.

For all non-zero values of m, the spherical average of $S_m(\theta, \phi)$ vanishes. So on taking averages in (6.17) and (6.18) we have

$$\left.\begin{aligned} \bar{\eta}_2 &= X_0 \cos 2\sigma_n t + Y_0, \\ \bar{\Phi}_2 &= Z_0 \frac{a}{r} \sin 2\sigma_n t, \end{aligned}\right\} \tag{6.20}$$

so that $\Phi_2 \sim \bar{\Phi}_2$. Moreover, to determine $\bar{\Phi}_2$ we may go directly to (5.14) and (5.15) for h_2 and $\bar{\Phi}_2$.

The mean values on the right of (5.14) and (5.15) may be evaluated by use of the following Lemmas:

LEMMA A

$$\overline{S_n^2} = \frac{1}{2n+1}. \tag{6.21}$$

LEMMA B

$$\overline{S_{n\theta}^2 + \frac{S_{n\phi}^2}{\sin^2\theta}} = \frac{n(n+1)}{2n+1}. \tag{6.22}$$

Hence from (5.14) we have

$$\left.\begin{aligned} h_{2t} - \bar{\Phi}_{2r} &= 0, \\ \bar{\Phi}_{2t} - \omega^2 a h_2 &= M \cos 2\sigma_n t + N, \end{aligned}\right\} \tag{6.23}$$

where
$$M = \frac{(n-1)(n+2)(4n-1)}{4(2n+1)} \frac{T}{a^3} A_n^2 \tag{6.24}$$

and
$$N = -\frac{3(n-1)(n+2)}{4(2n+1)} \frac{T}{a^3} A_n^2. \tag{6.25}$$

Equations (6.23) are satisfied by taking

$$\left.\begin{aligned} h_2 &= X\cos 2\sigma_n t + Y, \\ \Phi_2 &= Z\frac{a}{r}\sin 2\sigma_n t, \end{aligned}\right\} \tag{6.26}$$

where†

$$Z = \frac{2\sigma_n M}{4\sigma_n^2 - \omega^2}, \quad X = \frac{Z}{2\sigma_n a}, \quad Y = -\frac{N}{\omega^2 a}. \tag{6.27}$$

7. Discussion

The pressure fluctuation at large distances r corresponding to the solution (6.26) is

$$p_2 = -2\sigma_n Z\frac{a}{r}\cos 2\sigma_n t, \tag{7.1}$$

where Z is given by (6.27). This represents a monopole source of sound with radian frequency $2\sigma_n$ – twice the fundamental frequency σ_n of the natural, asymmetric oscillation. At distances comparable with the wavelength of sound, the term $2\sigma_n t$ in (7.1) is to be replaced by $2\sigma_n(t-r/c)$ where c is the speed of sound.‡

From (7.1) and the expressions (6.24) and (6.27) for M and Z we see that the amplitude of the pressure field is proportional to A_n^2, the square of the first-order wave amplitude. It is also inversely proportional to $(4\sigma_n^2-\omega^2)$, so it becomes large when the doubled frequency $2\sigma_n$ approaches the fundamental frequency ω of the radial normal mode. We can regard the effect as a 'resonance' between the second-order driving force and the radial mode of oscillation.

What is the nature of the driving force? We can investigate this question by supposing the gas pressure p_0 in the bubble to be such that the radial mode frequency vanishes. It is clear from (6.26) and (6.27) that there is still a monopole radiation. Hence the oscillation is *not* due essentially to any second-order change in the volume of the bubble. That indeed must occur, but is only an accompanying feature of the resultant oscillation.

The answer may be found in the dynamical boundary condition (4.3) which suggests that the driving force comes from two sources:

(1) The Bernoulli term $\frac{1}{2}(\text{velocity})^2$ in the pressure. This clearly has two maxima and two minima in every period $2\pi/\sigma_n$ of the fundamental oscillation.

(2) A second-order contribution from the surface-tension term $T\nabla\cdot\boldsymbol{n}$, as seen more explicitly in (4.12). The additional terms, such as $\eta(\partial/\partial r)\,T\nabla\cdot\boldsymbol{n}$, arising from the Taylor expansion about $r=a$, must be regarded as artifacts of the method of solution rather than real physical effects. It appears that all the resulting terms happen to be in-phase with the Bernoulli term.

† The above results differ from the analysis given by Tsamopoulos & Brown (1983) in some significant respects. Their expressions (56) to (58) contain no term in the denominator corresponding to $(4\sigma_n^2-\omega^2)$. Moreover in the lowest-order terms, a factor η^{-1} (corresponding to our r^{-1}) seems to have been omitted. Comparison with our results suggests that the authors may have made an assumption equivalent to setting $\omega=0$.

‡ When compressibility is taken into account, the linear theory also yields a small monopole component; but this is negligible compared with (7.1), unless A_n/a is very small.

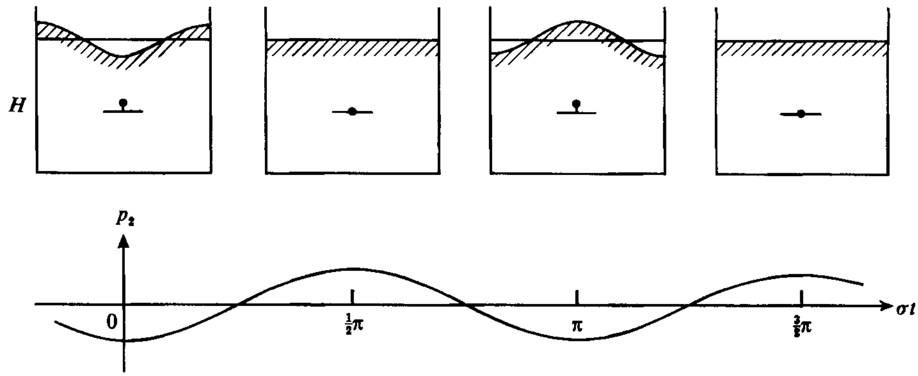

FIGURE 5. A standing wave on water of depth H, and the corresponding fluctuation p_2 in the mean pressure on the bottom.

8. Standing waves on deep water

An illuminating analogy exists between the nonlinear emission of sound by the distorted bubble oscillations, as discussed above, and the second-order production of underwater sound and microseisms by standing waves on deep water (Longuet-Higgins 1948, 1950, 1953).

Consider a simple surface wave on water of mean depth H as in figure 5, and suppose that the surface elevation η is given, to first order, by

$$\eta = A\cos(kx-\sigma t)+A'\cos(kx+\sigma t). \tag{8.1}$$

This represents two progressive waves of the same length $2\pi/k$ and radian frequency σ travelling in opposite senses. If $A' = A$ then we have a standing wave

$$\eta = 2A\cos kx\cos\sigma t. \tag{8.2}$$

According to linear theory, the first-order pressure fluctuations decay rapidly with the depth z below the surface, in fact like $\cosh k(z-H)$. Hence if z exceeds more than half a wavelength these fluctuations become negligible, just as the pressure fluctuations in an asymmetric bubble oscillation, at several 'wavelengths' $2\pi a/n$ from the surface of the bubble.

At second order, however, we find at some distance below the standing wave a pressure fluctuation

$$p_2 = -2A^2\sigma^2\cos 2\sigma t \tag{8.3}$$

which is proportional to the square of the wave amplitude, and is of twice the fundamental frequency σ. This term is *not* attenuated with the depth z. Its existence can be very simply related to the raising and lowering of the centre of gravity of the fluid, which occurs twice in a complete cycle (see Longuet-Higgins 1953).

In the general case (8.1) a similar argument shows that

$$p_2 = -2AA'\sigma^2\cos 2\sigma_n t, \tag{8.4}$$

so that if either A or A' vanishes, giving a pure progressive wave, then the pressure fluctuations vanish. This corresponds to the fact that in a progressive wave the centre of mass remains at a level constant in time. The formula (8.4) has been very well verified by simple laboratory experiments (see Cooper & Longuet-Higgins 1951).

The analogy becomes closer when we introduce the compressibility of the water. It is then found that the pressure fluctuation (8.3) gives rise to a sound wave in which

$$p_2 \sim -2A^2\sigma^2 \frac{\cos 2\sigma(z-H)/c}{\cos 2\sigma H/c} \cos 2\sigma t \tag{8.5}$$

propagated between the free surface $z = 0$ and the bottom $z = H$ (see Longuet-Higgins 1950, Section 4). When the depth H is equal to $(\frac{1}{2}n + \frac{1}{4})$ sound wavelengths we have a resonance condition when the denominator in (8.5) goes to zero and the pressure amplitude increases within limit, damping being neglected.

Lastly, when the compressibility of the bottom is also taken into account we find that the amplitude remains finite owing to radiation of seismic energy either vertically downwards, when the surface waves extend indefinitely in a horizontal sense, or horizontally as Stoneley waves, when the generating area is bounded. However, the resonance effect can still be seen in the response curve of the vertical displacement as a function of the frequency (Longuet-Higgins 1950, 1953).

For ocean surface waves, this mechanism was observed to be effective for microseisms in the range 3 to 8 s (Haubrich, Munk & Snodgrass 1963) which correspond to surface waves of period 6 to 16 s. An even more detailed agreement has recently been demonstrated by Kibblewhite (1987) for sea waves and microseisms near New Zealand. We note that Brekhovskikh (1966) suggested the same mechanism as a possible source of underwater sound at much higher frequencies, but in that context the sound is insignificant. Although high-frequency ripples may indeed exist on the ocean surface, it appears that these are not efficient sound generators. On the contrary, the bubble oscillations are very effective sound generators, as we shall see.

The inverse phenomenon, whereby standing waves on water can be generated subharmonically by the application of an external pressure (or a vertical acceleration) to the body of the fluid, has been known since Faraday (1931); see also Longuet-Higgins (1983). A theoretical analysis was given by Benjamin & Ursell (1954).

In a similar way, the generation of second-order monopole pressure by distorting bubbles may be viewed as the inverse of the well-known phenomenon whereby violent bubble distortions are produced subharmonically by applying a high-frequency pressure to the fluid (see Kornfeld & Suvorov 1944; Strasberg 1958; Benjamin 1958, 1964; Eller & Crum 1970).

9. Progressive bubble oscillations

The analogy with surface waves suggests that we consider other forms of perturbation of a bubble surface, particularly 'progressive waves', in which the surface displacement is given to first order by

$$\eta_1 = A\cos(m\phi - \sigma t)\, P_n^m(\cos\theta), \tag{9.1}$$

where $P_n^m(\cos\theta)$ denotes the associated Legendre polynomial:

$$P_n^m(\cos\theta) = \sin^m\theta \frac{\mathrm{d}^m}{\mathrm{d}(\cos\theta)^m} P_n(\cos\theta), \tag{9.2}$$

and $0 < m \leqslant n$. The expression (9.1) represents a wave of equatorial wavenumber m progressing eastwards around the sphere. Does such a wave produce monopole pressure fluctuations at second order?

Corresponding to (9.1) we have a first-order velocity potential

$$\Phi_1 = -\frac{a\sigma_n}{n+1} A \left(\frac{a}{r}\right)^{n+1} \sin(m\phi - \sigma_n t) P_n^m(\cos\theta). \tag{9.3}$$

On substituting the expression into the boundary conditions (5.6) and (5.7) we find that all the product terms on the right are functions of $(m\phi - \sigma_n t)$. Hence on averaging with respect to ϕ all the time-dependence vanishes. Therefore on the right of the averaged equations (5.12) and (5.13) there are no time-dependent terms, and the monopole terms vanish identically.

There will of course remain some constant terms, proportional to A^2. In an initial-value problem these can still give rise to a transient response proportional to $A^2 \cos \omega t$, which will decay under viscous and radiation damping.

Now suppose we consider the more general perturbation

$$\eta_1 = [A\cos(m\phi - \sigma_n t) + A' \cos(m\phi + \sigma_n t)] P_n^m(\cos\theta), \tag{9.4}$$

with

$$\Phi_1 = [-A\sin(m\phi - \sigma_n t) + A' \sin(m\phi + \sigma_n t)] \frac{a\sigma_n}{n+1} \left(\frac{a}{r}\right)^{n+1} P_n^m(\cos\theta). \tag{9.5}$$

This represents the superposition of two waves of the same wavenumber m/a and frequency σ_n, travelling in opposite directions. On the right-hand sides of (5.12) and (5.13) there will now be three groups of terms, proportional to A^2, AA' and A'^2 respectively. The terms in A^2 will be functions of $(m\phi - \sigma t)$ and so on averaging will make no time-dependent contribution. The terms in A'^2 will be functions of $(m\phi + \sigma t)$ and will make no contribution similarly. The terms in AA', however, will make a contribution. The factor multiplying AA' will be the same as if A and A' were both equal, that is to say as if

$$\eta_1 = 2A \cos m\phi \, P_n^m(\cos\theta) \cos \sigma_n t \tag{9.6}$$

and

$$\Phi_1 = 2A \frac{a\sigma_n}{n+1} \left(\frac{a}{r}\right)^{n+1} \cos m\phi \, P_n^m(\cos\theta) \sin \sigma_n t. \tag{9.7}$$

Writing

$$P_n^m(\cos\theta) \cos m\phi = S_n(\theta, \phi) \tag{9.8}$$

the expressions on the right of (9.6) and (9.7) are formally the same as in (6.1), (6.2) and (6.6), but with $A = \frac{1}{2}A_n$. Accordingly the analysis of §§6 and 7 applies precisely, though with different normalizing constants in (6.21) and (6.22). Hence we have a monopole radiation proportional to the product AA', an exact analogue to the pressure fluctuations from standing waves in deep water (equation (8.4)).

The terms in A^2, AA' and A'^2 will also yield a constant, second-order change in the pressure near the bubble surface, which must be taken into account in initial-value problems.

10. Discussion and conclusions

We have shown that the normal-mode distortions of bubbles will emit a monopole radiation nonlinearly, having a frequency double that of the basic frequency. This effect is precisely analogous to the generation of unattenuated pressure oscillations, at second order, beneath standing surface waves on deep water, and the consequent radiation of sound or microseisms. In the present case the free surface is the surface of the bubble rather than the horizontal surface of the sea.

The order of magnitude of the second-order pressure fluctuations near the surface of the bubble appears to be large enough to invite further investigation as a possible source of underwater sound in the ocean, and elsewhere. In a second paper (Longuet-Higgins 1989) we shall consider the underwater sound signal produced, at second order, by an arbitrary initial distortion of a bubble, and make detailed calculations to be compared with observations.

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